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Reduction of Jacobi–Nijenhuis manifolds

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Abstract

A reduction theorem for Jacobi–Nijenhuis manifolds is established and its relation with the reduction of homogeneous Poisson–Nijenhuis structures is shown. Reduction under Lie group actions is also studied. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of Jacobi–Nijenhuis structure was introduced by Marrero et al. [7]. Recently, the authors gave, in [13], a more strict definition of that structure which generalises, in a natural way, the notion of Poisson–Nijenhuis manifold introduced by Magri and Morosi [3,6] for better understanding the completely integrable hamiltonian systems.

In this paper, we intend to study the reduction of Jacobi–Nijenhuis structures. Mainly, we define a foliation on a submanifold of a Jacobi–Nijenhuis manifold in such a way that the manifold of the leaves is also endowed with a Jacobi–Nijenhuis structure. Since a Jacobi–Nijenhuis manifold carries a Jacobi structure and, on the other hand, there is a close relation between Jacobi–Nijenhuis manifolds and homogeneous Poisson–Nijenhuis manifolds, we were inspired in some technical arguments used in the reduction methods of both Jacobi [9,10] and Poisson–Nijenhuis manifolds [14], in order to achieve our goal.

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This paper is organised as follows. In Section 2, we review some basic facts about Jacobi manifolds, including the reduction method. In Section 3, we give a reduction theorem for homogeneous Poisson–Nijenhuis manifolds, which is adapted from the Poisson–Nijenhuis reduction theorem of Vaisman [14]. Section 4 is devoted to Jacobi–Nijenhuis manifolds. We recall the essential definitions and the notions of associated homogeneous Poisson–Nijenhuis manifold and conformal equivalence. In Section 5, we establish a reduction theorem for Jacobi–Nijenhuis manifolds, we study the reduction of conformally equivalent Jacobi–Nijenhuis structures and we show how the homogeneous Poisson–Nijenhuis reduction is related with the Jacobi–Nijenhuis reduction. Section 6 concerns the reduction of Jacobi–Nijenhuis structures under Lie group actions. The two cases presented are examples of the reduction theorem of previous section. In the first case, we obtain a Jacobi–Nijenhuis structure on the space of the orbits of a Lie group action. In the second, the action has a momentum map and the Jacobi–Nijenhuis structure is defined on a quotient of a level set of that momentum map.

Notation: In the following, we will denote by M a C^∞-differentiable manifold of finite dimension, by $C^\infty(M)$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$, $k \in \mathbb{N}$, the space of k-forms on M, and by $\mathcal{V}^k(M)$, $k \in \mathbb{N}$, the space of skew-symmetric contravariant k-tensors on M.

2. Jacobi manifolds

We consider the manifold M endowed with a 2-tensor Λ and a vector field E. The following bracket on $C^\infty(M)$,

$$\{f, g\} = \Lambda(\mathrm{d}f, \mathrm{d}g) + \langle f\,\mathrm{d}g - g\,\mathrm{d}f, E\rangle, \quad f, g \in C^\infty(M), \tag{1}$$

is bilinear and skew-symmetric, and satisfies the Jacobi identity if and only if

$$[\Lambda, \Lambda] = -2E \wedge \Lambda \quad \text{and} \quad [E, \Lambda] = 0, \tag{2}$$

where $[\,,\,]$ denotes the Schouten bracket [4]. When conditions (2) are verified, the pair (Λ, E) defines a *Jacobi structure* on M and (M, Λ, E) is called a *Jacobi manifold*. The bracket (1) is the *Jacobi bracket* and $(C^\infty(M), \{\,,\,\})$ is a local Lie algebra in the sense of Kirillov (cf. [2]). If the vector field E identically vanishes on M, conditions (2) reduce to $[\Lambda, \Lambda] = 0$, and M is endowed with a *Poisson structure*.

We denote by $\Lambda^\# : T^*M \to TM$ and $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ the vector bundle maps associated with Λ and (Λ, E), respectively; i.e., for all α, β sections of T^*M and $f \in C^\infty(M)$,

$$\langle \beta, \Lambda^\#(\alpha)\rangle = \Lambda(\alpha, \beta) \tag{3}$$

and

$$(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E\rangle). \tag{4}$$

These vector bundle maps can be considered as homomorphisms of $C^\infty(M)$-modules, $\Lambda^\# : \Omega^1(M) \to \mathcal{V}^1(M)$ and $(\Lambda, E)^\# : \Omega^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$, respectively.

For any $f \in C^\infty(M)$, the vector field on M

$$X_f = \Lambda^\#(\mathrm{d}f) + fE, \tag{5}$$

is called the *hamiltonian vector field* associated with f.

The space $\Omega^1(M) \times C^\infty(M)$ possesses a Lie algebra structure whose bracket $\{,\}$ is defined as follows (cf. [1]): for all $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M)$,

$$\{(\alpha, f), (\beta, g)\} := (\gamma, h), \tag{6}$$

where

$$\gamma := L_{\Lambda^\#(\alpha)}\beta - L_{\Lambda^\#(\beta)}\alpha - \mathrm{d}(\Lambda(\alpha, \beta)) + fL_E\beta - gL_E\alpha - i_E(\alpha \wedge \beta),$$
$$h := -\Lambda(\alpha, \beta) + \Lambda(\alpha, \mathrm{d}g) - \Lambda(\beta, \mathrm{d}f) + \langle f\,\mathrm{d}g - g\,\mathrm{d}f, E\rangle,$$

(L is the Lie derivative operator).

Let $a \in C^\infty(M)$ be a function which vanishes nowhere on M. For all $f, g \in C^\infty(M)$, we may define

$$\{f, g\}^a := \frac{1}{a}\{af, ag\}. \tag{7}$$

This new bracket $\{,\}^a$ on $C^\infty(M)$ defines another Jacobi structure (Λ^a, E^a) on M, which is said to be *a-conformal* to the initially given one. The two Jacobi structures (Λ, E) and (Λ^a, E^a) are said to be *conformally equivalent* and

$$\Lambda^a = a\Lambda, \qquad E^a = \Lambda^\#(\mathrm{d}a) + aE. \tag{8}$$

A *homogeneous Poisson manifold* (M, Λ, T) is a Poisson manifold (M, Λ) with a vector field $T \in \mathcal{V}^1(M)$ such that

$$L_T\Lambda = [T, \Lambda] = -\Lambda. \tag{9}$$

Homogeneous Poisson manifolds are closely related to Jacobi manifolds. With each Jacobi manifold (M, Λ, E) we may associate a homogeneous Poisson manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$, with

$$\tilde{M} = M \times \mathbb{R}, \qquad \tilde{\Lambda} = \mathrm{e}^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right) \quad \text{and} \quad \tilde{T} = \frac{\partial}{\partial t}, \tag{10}$$

where t is the usual coordinate on $\mathbb{R}$ [5]. The manifold $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ is called the *Poissonization* of (M, Λ, E).

Let us now recall the reduction procedure for Jacobi manifolds.

Theorem 2.1 (Mikami [9] and Nunes da Costa [10]). *Let (M, Λ, E) be a Jacobi manifold, S a submanifold of M and F a vector sub-bundle of T_SM, which satisfy the following conditions*:

1. *the distribution $TS \cap F$ is completely integrable and the foliation of S defined by this distribution is simple, i.e., all the leaves have the same dimension and the set $\hat{S}$ of leaves has the structure of a differentiable manifold for which the canonical projection $\pi : S \to \hat{S}$ is a submersion*;

2. *for any* $f, h \in C^\infty(M)$ *with differentials* $\mathrm{d}f$ *and* $\mathrm{d}h$*, restricted to* S*, vanishing on* F*, the differential* $\mathrm{d}\{f, h\}$*, restricted to* S*, vanishes on* F*;*
3. *if* $F^0 \subset T^*_S M$ *denotes the annihilator of* F*, then* $(\Lambda|_S)^\#(F^0) \subset TS + F$*, and the restriction of* E *to* S *is a differentiable section of* $TS + F$*.*

Then, there exists on $\hat{S}$ *a unique Jacobi structure* $(\hat{\Lambda}, \hat{E})$ *whose associated bracket is given, for any* $\hat{f}, \hat{h} \in C^\infty(\hat{S})$ *and any differentiable extensions* f *of* $\hat{f} \circ \pi$ *and* h *of* $\hat{h} \circ \pi$ *with differentials* $\mathrm{d}f$ *and* $\mathrm{d}h$*, restricted to* S*, vanish on* F*, by*

$$\{\hat{f}, \hat{h}\} \circ \pi = \{f, h\} \circ i, \tag{11}$$

where i *is the canonical injection of* S *in* M*.*

The Jacobi manifold $(\hat{S}, \hat{\Lambda}, \hat{E})$ *is said to have been obtained from* (M, Λ, E) *by reduction via* (S, F)*.*

Let $\lambda : T_S M \to TS$ be a (projection) vector bundle map such that its restriction to TS is the identity map and $F \subset \mathit{Ker}\,\lambda$. Then, the Jacobi structures (Λ, E) on M and $(\hat{\Lambda}, \hat{E})$ on $\hat{S}$ are related by the formulae:

$$\hat{\Lambda}^\#_{\pi(x)} = T_x\pi \circ \lambda_x \circ \Lambda^\#_{i(x)} \circ {}^{\mathrm{t}}\lambda_x \circ {}^{\mathrm{t}}T_x\pi, \quad x \in S, \tag{12}$$

$$\hat{E} \circ \pi = T\pi \circ \lambda \circ E \circ i. \tag{13}$$

We remark that the transpose of λ, ${}^{\mathrm{t}}\lambda : T^*S \to T^*_S M$, is the injection that extends each linear form on S to a linear form on M that vanishes on $\mathit{Ker}\,\lambda$.

3. Reduction of homogeneous Poisson–Nijenhuis manifolds

This section is devoted to Poisson–Nijenhuis and homogeneous Poisson–Nijenhuis manifolds. We give a reduction theorem for homogeneous Poisson–Nijenhuis manifolds.

A *Nijenhuis operator* on a differentiable manifold M is a tensor field N of type $(1, 1)$ which has a vanishing Nijenhuis torsion:

$$T(N)(X, Z) = [NX, NZ] - N[NX, Z] - N[X, NZ] + N^2[X, Z] = 0,$$
$$X, Z \in \mathcal{V}^1(M).$$

A *Poisson–Nijenhuis manifold* (M, Λ_0, N) is a Poisson manifold (M, Λ_0) with a Nijenhuis tensor N which is compatible with Λ_0, i.e.: (i) $N\Lambda_0^\# = \Lambda_0^\# \, {}^{\mathrm{t}}N$, where ${}^{\mathrm{t}}N$ is the transpose of N, and (ii) the map $\Lambda_0^\# \circ C(\Lambda_0, N) : \Omega^1(M) \times \Omega^1(M) \to \mathcal{V}^1(M)$ identically vanishes on M. $C(\Lambda_0, N)$ is the *Magri–Morosi concomitant* of Λ_0 and N [6] defined, for all $(\alpha, \beta) \in \Omega^1(M) \times \Omega^1(M)$, by

$$C(\Lambda_0, N)(\alpha, \beta) = \{\alpha, \beta\}_1 - \{{}^{\mathrm{t}}N\alpha, \beta\}_0 - \{\alpha, {}^{\mathrm{t}}N\beta\}_0 + {}^{\mathrm{t}}N\{\alpha, \beta\}_0, \tag{14}$$

where $\{,\}_i$ is the bracket associated with Λ_i, $\Lambda_i^\# = N^i \Lambda_0^\#$, $i = 0, 1$, that defines a Lie algebra structure on $\Omega^1(M)$ [3]. N is called the *recursion operator* of (M, Λ_0, N).

In what concerns the reduction procedure, remark that, when a Jacobi manifold is Poisson, Theorem 2.1 is the Marsden–Ratiu Poisson reduction theorem [8]. This last one was refined by Vaisman [14] in order to include the Poisson–Nijenhuis case.

Theorem 3.1 (Vaisman [14]). *Let (M, Λ, N) be a Poisson–Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ verifying conditions 1 and 2 of Theorem* 2.1.[3] *Moreover, if $N|_S(TS) \subset TS$, $N|_S(F) \subset F$, $N|_S$ sends projectable vector fields to projectable vector fields, and $(\Lambda|_S)^\#(F^0) \subset TS$, then there exists on $\hat{S}$ a Poisson–Nijenhuis structure $(\hat{\Lambda}, \hat{N})$, obtained from (Λ, N) by reduction via (S, F).*

The Poisson tensor $\hat{\Lambda}$ on $\hat{S}$ is associated with the vector bundle map $\hat{\Lambda}^\#$ given by (12) and the tensor $\hat{N}$ of type $(1, 1)$ on $\hat{S}$ is given by

$$\hat{N} = T\pi \circ \lambda \circ N|_S \circ \lambda_h^{-1} \circ (T\pi)_h^{-1}, \tag{15}$$

where λ_h is the restriction of λ to $TS \subset T_S M$, which is the identity map, and $(T\pi)_h$ is the restriction of $T\pi$ to the horizontal vector sub-bundle of TS with respect to the decomposition $TS \equiv T\hat{S} \oplus (TS \cap F)$.

Let us introduce a tensor field N_S of type $(1, 1)$ on the submanifold S by setting

$$N_S = \lambda \circ N|_S \circ \lambda_h^{-1}. \tag{16}$$

Then (15) can be written as

$$\hat{N} = T\pi \circ N_S \circ (T\pi)_h^{-1}. \tag{17}$$

Definition 3.2. A homogeneous Poisson–Nijenhuis manifold (M, Λ, N, T) is a Poisson–Nijenhuis manifold (M, Λ, N) with a vector field $T \in \mathcal{V}^1(M)$ such that

$$L_T \Lambda = -\Lambda \quad \text{and} \quad L_T N = 0. \tag{18}$$

Remark 3.3. Conditions (18) assure that, for all $k \in \mathbb{N}$, $L_T \Lambda_k = -\Lambda_k$, where Λ_k is the Poisson tensor associated with $\Lambda_k^\# = N^k \Lambda$. That is, all the members of the hierarchy $(\Lambda_k, k \in \mathbb{N})$ are homogeneous Poisson tensors on M with respect to the vector field T.

Theorem 3.1 can easily be adapted to include homogeneous Poisson–Nijenhuis reduction case.

Theorem 3.4. *Let (M, Λ, N, T) be a homogeneous Poisson–Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ such that all the conditions of Theorem* 3.1 *are verified, and denote by $(\hat{S}, \hat{\Lambda}, \hat{N})$ the Poisson–Nijenhuis manifold obtained from (M, Λ, N) by reduction via (S, F). If the vector field $T \in \mathcal{V}^1(M)$ is tangent to S, $T|_S \notin$ Ker λ and $\lambda(T|_S) = T_S \in \mathcal{V}^1(S)$ is a projectable vector field with projection $\hat{T} \in \mathcal{V}^1(\hat{S})$, then $(\hat{S}, \hat{\Lambda}, \hat{N}, \hat{T})$ is a homogeneous Poisson–Nijenhuis manifold.*

Proof. We only have to prove that $L_{\hat{T}} \hat{\Lambda} = -\hat{\Lambda}$ and $L_{\hat{T}} \hat{N} = 0$.

It is easy to verify that the tensor field $L_{T_S} N_S$ on S is projectable and its projection is $L_{\hat{T}} \hat{N}$, i.e.,

$$L_{\hat{T}} \hat{N} = T\pi \circ L_{T_S} N_S \circ (T\pi)_h^{-1}, \tag{19}$$

[3] Obviously, the bracket considered in condition 2 of Theorem 2.1 is the Poisson bracket on (M, Λ).

where N_S is given by (16). Since T is tangent to S and $N|_S(TS) \subset TS$, (19) can be written as

$$L_{\hat{T}}\hat{N} = T\pi \circ \lambda \circ (L_{T|_S}N|_S) \circ \lambda_h^{-1} \circ (T\pi)_h^{-1}. \tag{20}$$

Taking into account that $L_T N = 0$, from (20) we obtain $L_{\hat{T}}\hat{N} = 0$.

On the other hand, for all $\hat{\alpha}, \hat{\beta} \in \Omega^1(\hat{S})$,

$$(L_{T|_S}\Lambda|_S)({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}), {}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\beta})) = -\Lambda|_S({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}), {}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\beta})). \tag{21}$$

The second member of (21) equals $-\hat{\Lambda}(\hat{\alpha}, \hat{\beta})$. Using the facts that T is tangent to S, the two 1-forms ${}^{\mathrm{t}}\lambda(L_{T_S}({}^{\mathrm{t}}T\pi(\hat{\alpha})))$ and $L_{T|_S}({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}))$ coincide on TS, and $(\Lambda|_S)^{\#}((TS)^0) \subset F$, we conclude that the first member of (21) equals $(L_{\hat{T}}\hat{\Lambda})(\hat{\alpha}, \hat{\beta})$. So, $L_{\hat{T}}\hat{\Lambda} = -\hat{\Lambda}$, because $\hat{\alpha}$ and $\hat{\beta}$ are arbitrary. □

4. Jacobi–Nijenhuis manifolds

The initial definition of Jacobi–Nijenhuis manifold was introduced by Marrero et al. [7]. In [13], the authors gave a more strict definition of this concept. In this section, we review the essential results concerning this structure needed throughout this article.

Let M be a C^∞-differentiable manifold and $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$, a $C^\infty(M)$-linear map defined, for all $(X, f) \in \mathcal{V}^1(M) \times C^\infty(M)$, by

$$\mathcal{N}(X, f) = (NX + fY, \langle \gamma, X\rangle + gf), \tag{22}$$

where N is a tensor field of type $(1, 1)$ on M, $Y \in \mathcal{V}^1(M)$, $\gamma \in \Omega^1(M)$ and $g \in C^\infty(M)$. $\mathcal{N} := (N, Y, \gamma, g)$ can be considered as a vector bundle map, $\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}$. Since the space $\mathcal{V}^1(M) \times C^\infty(M)$ endowed with the bracket

$$[(X, f), (Z, h)] = ([X, Z], X \cdot h - Z \cdot f), \tag{23}$$

$((X, f), (Z, h)) \in (\mathcal{V}^1(M) \times C^\infty(M, \mathbf{R}))^2$, is a real Lie algebra, we may define the *Nijenhuis torsion* $\mathcal{T}(\mathcal{N})$ of $\mathcal{N}$. It is a $C^\infty(M)$-bilinear map $\mathcal{T}(\mathcal{N}) : (\mathcal{V}^1(M) \times C^\infty(M))^2 \to \mathcal{V}^1(M) \times C^\infty(M)$ given by

$$\begin{aligned}\mathcal{T}(\mathcal{N})((X, f), (Z, h)) = {}& [\mathcal{N}(X, f), \mathcal{N}(Z, h)] - \mathcal{N}[\mathcal{N}(X, f), (Z, h)] \\ & - \mathcal{N}[(X, f), \mathcal{N}(Z, h)] + \mathcal{N}^2[(X, f), (Z, h)],\end{aligned} \tag{24}$$

$((X, f), (Z, h)) \in (\mathcal{V}^1(M) \times C^\infty(M))^2$.

Definition 4.1. A $C^\infty(M)$-linear map $\mathcal{N} : \mathcal{V}^1(M) \times C^\infty(M) \to \mathcal{V}^1(M) \times C^\infty(M)$ is a Nijenhuis operator on M, if it has a vanishing Nijenhuis torsion.

Suppose now that M is equipped with a Jacobi structure (Λ_0, E_0) and a Nijenhuis operator $\mathcal{N}$. Then, we may define a tensor field Λ_1 of type $(2, 0)$ and a vector field E_1 on M, by setting

$$(\Lambda_1, E_1)^{\#} = \mathcal{N} \circ (\Lambda_0, E_0)^{\#}. \tag{25}$$

Recall that two Jacobi structures (Λ_0, E_0) and (Λ_1, E_1), defined on the same differentiable manifold, are said to be *compatible* if their sum $(\Lambda_0 + \Lambda_1, E_0 + E_1)$ is again a Jacobi structure (cf. [12]).

If one looks for the conditions that assure the pair (Λ_1, E_1), given by (25), defines a new Jacobi structure on M, compatible with (Λ_0, E_0), one finds (cf. [7]):

1. Λ_1 is skew-symmetric if and only if $\mathcal{N} \circ (\Lambda_0, E_0)^{\#} = (\Lambda_0, E_0)^{\#} \circ {}^{\mathrm{t}}\mathcal{N}$, where ${}^{\mathrm{t}}\mathcal{N}$ is the transpose of $\mathcal{N}$. This condition is equivalent to $NE_0 = \Lambda_0^{\#}(\gamma) + gE_0$, $N\Lambda_0^{\#} - Y \otimes E_0 = \Lambda_0^{\#\mathrm{t}}N + E_0 \otimes Y$ and $\langle \gamma, E_0 \rangle = 0$. Then,
$$\Lambda_1^{\#} = N\Lambda_0^{\#} - Y \otimes E_0 = \Lambda_0^{\#\mathrm{t}}N + E_0 \otimes Y \tag{26}$$
and
$$E_1 = NE_0 = \Lambda_0^{\#}(\gamma) + gE_0. \tag{27}$$
2. When Λ_1 is skew-symmetric, (Λ_1, E_1) defines a Jacobi structure on M if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^{\infty}(M)$,
$$\begin{aligned}&\mathcal{T}(\mathcal{N})((\Lambda_0, E_0)^{\#}(\alpha, f), (\Lambda_0, E_0)^{\#}(\beta, h))\\ &\quad = \mathcal{N} \circ (\Lambda_0, E_0)^{\#}(\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))),\end{aligned}$$
where $\mathcal{C}((\Lambda_0, E_0), \mathcal{N})$ is the *concomitant* of (Λ_0, E_0) and $\mathcal{N}$ which is given, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^{\infty}(M)$, by
$$\begin{aligned}&\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))\\ &\quad = \{(\alpha, f), (\beta, h)\}_1 - \{{}^{\mathrm{t}}\mathcal{N}(\alpha, f), (\beta, h)\}_0\\ &\qquad - \{(\alpha, f), {}^{\mathrm{t}}\mathcal{N}(\beta, h)\}_0 + {}^{\mathrm{t}}\mathcal{N}\{(\alpha, f), (\beta, h)\}_0,\end{aligned}$$
($\{,\}_i$ is the bracket (6) associated with the Jacobi structure (Λ_i, E_i), $i = 0, 1$).
3. In the case where (Λ_1, E_1) is a Jacobi structure, it is compatible with (Λ_0, E_0) if and only if, for all $(\alpha, f), (\beta, h) \in \Omega^1(M) \times C^{\infty}(M)$,
$$(\Lambda_0, E_0)^{\#}(\mathcal{C}((\Lambda_0, E_0), \mathcal{N})((\alpha, f), (\beta, h))) = 0.$$

Definition 4.2. A Jacobi–Nijenhuis manifold $(M, (\Lambda_0, E_0), \mathcal{N})$ is a Jacobi manifold (M, Λ_0, E_0) with a Nijenhuis operator $\mathcal{N}$ which is compatible with (Λ_0, E_0), i.e.: (i) $\mathcal{N} \circ (\Lambda_0, E_0)^{\#} = (\Lambda_0, E_0)^{\#} \circ {}^{\mathrm{t}}\mathcal{N}$ and (ii) the map $(\Lambda_0, E_0)^{\#} \circ \mathcal{C}((\Lambda_0, E_0), \mathcal{N}) : (\Omega^1(M) \times C^{\infty}(M))^2 \to \mathcal{V}^1(M) \times C^{\infty}(M)$ identically vanishes on M. $\mathcal{N}$ is called the recursion operator of $(M, (\Lambda_0, E_0), \mathcal{N})$.

Theorem 4.3 (Marrero et al. [7]). *Let $((\Lambda_0, E_0), \mathcal{N})$ be a Jacobi–Nijenhuis structure on a differentiable manifold M. Then, there exists a hierarchy $((\Lambda_k, E_k), k \in \mathbb{N})$ of Jacobi structures on M, which are pairwise compatible. For all $k \in \mathbb{N}$, (Λ_k, E_k) is the Jacobi structure associated with the vector bundle map $(\Lambda_k, E_k)^{\#}$ given by $(\Lambda_k, E_k)^{\#} = \mathcal{N}^k \circ (\Lambda_0, E_0)^{\#}$. Moreover, for all $k, l \in \mathbb{N}$, the pair $((\Lambda_k, E_k), \mathcal{N}^l)$ defines a Jacobi–Nijenhuis structure on M.*

Next proposition shows a relation between Jacobi–Nijenhuis manifolds and homogeneous Poisson–Nijenhuis structures.

Proposition 4.4 (Petalidou and Nunes da Costa [13]). *With each Jacobi–Nijenhuis manifold* $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, *a homogeneous Poisson–Nijenhuis manifold* $(\tilde{M}, \tilde{\Lambda}, \tilde{N}, \tilde{T})$ *can be associated, where* $(\tilde{M}, \tilde{\Lambda}, \tilde{T})$ *is the Poissonization of* (M, Λ, E) *and* $\tilde{N}$ *is the Nijenhuis tensor field on* $\tilde{M}$ *given by*

$$\tilde{N} = N + Y \otimes \mathrm{d}t + \frac{\partial}{\partial t} \otimes \gamma + g \frac{\partial}{\partial t} \otimes \mathrm{d}t. \tag{28}$$

Finally, we recall the notion of conformal equivalence of Jacobi–Nijenhuis structures on a differentiable manifold M.

Proposition 4.5 (Petalidou and Nunes da Costa [13]). *Let* $((\Lambda_0, E_0), \mathcal{N})$ *be a Jacobi–Nijenhuis structure on* M, (Λ_1, E_1) *the Jacobi structure associated with* $(\Lambda_1, E_1)^\# = \mathcal{N} \circ (\Lambda_0, E_0)^\#$, $a \in C^\infty(M)$ *a function which vanishes nowhere, and* (Λ_0^a, E_0^a) (*resp.* (Λ_1^a, E_1^a)) *the Jacobi structure* a*-conformal to* (Λ_0, E_0) (*resp.* (Λ_1, E_1)). *Then, there exists a Nijenhuis operator* $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ *such that* $(\Lambda_1^a, E_1^a)^\# = \mathcal{N}^a \circ (\Lambda_0^a, E_0^a)^\#$, *with*

$$N^a = N - Y \otimes \frac{\mathrm{d}a}{a}, \qquad Y^a = Y, \tag{29}$$

$$\gamma^a = \gamma + {}^{\mathrm{t}}N \frac{\mathrm{d}a}{a} - \left(g + \frac{1}{a} L_Y a \right) \frac{\mathrm{d}a}{a}, \qquad g^a = g + \frac{1}{a} L_Y a. \tag{30}$$

The Jacobi–Nijenhuis structure $((\Lambda_0^a, E_0^a), \mathcal{N}^a)$ *is said to be a-conformal to* $((\Lambda_0, E_0), \mathcal{N})$.

5. Reduction of Jacobi–Nijenhuis manifolds

In this section, we present the main result of this paper: a reduction theorem for Jacobi–Nijenhuis manifolds. We also study the reduction of conformally equivalent Jacobi–Nijenhuis structures and the relation between the Jacobi–Nijenhuis and homogeneous Poisson–Nijenhuis reduction.

Theorem 5.1. *Let* $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, *be a Jacobi–Nijenhuis manifold,* S *a submanifold of* M, $i : S \hookrightarrow M$ *the canonical injection, and* F *a vector sub-bundle of* $T_S M$, *which satisfy the conditions 1 and 2 of Theorem* 2.1 *and also*

1. $(\Lambda|_S)^\#(F^0) \subset TS$ *and* $E|_S$ *is a section of* TS;
2. $N|_S(TS) \subset TS$, $N|_S(F) \subset F$ *and* N_S, *given by* (16), *sends projectable vector fields to projectable vector fields*;
3. Y *is tangent to* S *and* $Y_S = \lambda(Y|_S) \in \mathcal{V}^1(S)$ *is a projectable vector field, where* $\lambda : T_S M \to TS$ *is a (projection) vector bundle map such that its restriction to* TS *is the identity map and* $F \subset \mathrm{Ker}\, \lambda$;

4. $\gamma|_S$ *is a section of* $(TS \cap F)^0$ *and, for all sections* Z of $TS \cap F$, $i_Z d({}^{\mathrm{t}}(Ti)(\gamma|_S)) = 0$;
5. $g|_S$ *is constant on the leaves of* S.

Under these conditions, there exists on $\hat{S}$ *a Jacobi–Nijenhuis structure* $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, *where* $(\hat{\Lambda}, \hat{E})$ *is given by* (12) and (13), $\hat{N}$ *is given by* (17), $\hat{Y} = T\pi \circ \lambda \circ Y|_S$, $\hat{\gamma} \in \Omega^1(\hat{S})$ *is such that* ${}^{\mathrm{t}}T\pi(\hat{\gamma}) = {}^{\mathrm{t}}(Ti)(\gamma|_S)$, *and* $\hat{g} \in C^\infty(\hat{S})$ *is given by* $\hat{g} \circ \pi = g|_S$. *The Jacobi–Nijenhuis manifold* $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ *is said to have been obtained from* $(M, (\Lambda, E), \mathcal{N})$ *by reduction via* (S, F).

Proof. Since all the conditions of Theorem 2.1 hold, $\hat{S}$ is endowed with a (reduced) Jacobi structure $(\hat{\Lambda}, \hat{E})$, given by (12) and (13). It remains to show that the Nijenhuis operator $\mathcal{N} := (N, Y, \gamma, g)$ also reduces to a Nijenhuis operator $\hat{\mathcal{N}}$ on $\hat{S}$ compatible with $(\hat{\Lambda}, \hat{E})$.

As in the case of Theorem 3.1, condition 2 guarantees the existence of a tensor field $\hat{N}$ of type (1, 1) on $\hat{S}$, given by (17). From condition 3, the vector field $Y_S = \lambda(Y|_S) \in \mathcal{V}^1(S)$ is projectable and we denote by $\hat{Y} \in \mathcal{V}^1(\hat{S})$ its projection. Also, by hypothesis 4, the 1-form $\gamma_S = {}^{\mathrm{t}}(Ti)(\gamma|_S)$ on S is projectable and we denote by $\hat{\gamma} \in \Omega^1(\hat{S})$ its projection. Finally, from condition 5, there exists a function $\hat{g} \in C^\infty(\hat{S})$ such that $\hat{g} \circ \pi = g|_S$. Thus, we obtain a $C^\infty(\hat{S})$-linear map, $\hat{\mathcal{N}} : \mathcal{V}^1(\hat{S}) \times C^\infty(\hat{S}) \to \mathcal{V}^1(\hat{S}) \times C^\infty(\hat{S})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, defined as in (22). Using the properties of the restriction $\mathcal{N}|_S := (N|_S, Y|_S, \gamma|_S, g|_S)$ of $\mathcal{N}$ to the submanifold S, a straightforward calculation shows that $\hat{\mathcal{N}}$ has a vanishing Nijenhuis torsion.

In order to conclude that $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ defines a Jacobi–Nijenhuis structure on $\hat{S}$, we have to prove that $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^\# = (\hat{\Lambda}, \hat{E})^\# \circ {}^{\mathrm{t}}\hat{\mathcal{N}}$ and that $(\hat{\Lambda}, \hat{E})^\# \circ \mathcal{C}((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}}) = 0$. Let $\hat{\alpha} \in \Omega^1(\hat{S})$, $\hat{f} \in C^\infty(\hat{S})$, and consider ${}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha})$, which is a section of T_S^*M, and $f \in C^\infty(M)$ an extension of $\hat{f} \circ \pi$, i.e., $f|_S = \hat{f} \circ \pi$. Then,

$$\mathcal{N}|_S((\Lambda|_S, E|_S)^\#({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}), f|_S)) = (\Lambda|_S, E|_S)^\#({}^{\mathrm{t}}\mathcal{N}|_S({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}), f|_S)). \tag{31}$$

Since $(\Lambda|_S)^\#({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}))$ is a section of $(\Lambda|_S)^\#(F^0) \subset TS$ and $E|_S$ is a section of TS, the image by $(T\pi \circ \lambda)$ of the term vector field of the first member of (31) is equal to

$$\hat{N}(\hat{\Lambda}^\#(\hat{\alpha})) + \hat{f}\hat{N}(\hat{E}) - \langle \hat{\alpha}, \hat{E} \rangle \hat{Y}. \tag{32}$$

Because ${}^{\mathrm{t}}\lambda({}^{\mathrm{t}}N_S({}^{\mathrm{t}}T\pi(\hat{\alpha}))) - {}^{\mathrm{t}}N|_S({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha}))$ is a section of $(TS)^0$ and $(\Lambda|_S)^\#((TS)^0) \subset F$, we get

$$T\pi \circ \lambda((\Lambda|_S)^\#({}^{\mathrm{t}}N|_S({}^{\mathrm{t}}(T\pi \circ \lambda)(\hat{\alpha})))) = \hat{\Lambda}^\#({}^{\mathrm{t}}\hat{N}(\hat{\alpha})),$$

and we may conclude that the image by $(T\pi \circ \lambda)$ of the term vector field of the second member of (31) is equal to

$$\hat{\Lambda}^\#({}^{\mathrm{t}}\hat{N}(\hat{\alpha})) + \hat{f}\hat{\Lambda}^\#(\hat{\gamma}) + \langle \hat{\alpha}, \hat{Y} \rangle \hat{E} + \hat{f}\hat{g}\hat{E}. \tag{33}$$

From (32) and (33), we obtain

$$\hat{N}(\hat{\Lambda}^\#(\hat{\alpha})) + \hat{f}\hat{N}(\hat{E}) - \langle \hat{\alpha}, \hat{E} \rangle \hat{Y} = \hat{\Lambda}^\#({}^{\mathrm{t}}\hat{N}(\hat{\alpha})) + \hat{f}\hat{\Lambda}^\#(\hat{\gamma}) + \langle \hat{\alpha}, \hat{Y} \rangle \hat{E} + \hat{f}\hat{g}\hat{E},$$

which means that the term vector field of $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^\#(\hat{\alpha}, \hat{f})$ coincides with the term vector field of $(\hat{\Lambda}, \hat{E})^\# \circ {}^{\mathrm{t}}\hat{\mathcal{N}}(\hat{\alpha}, \hat{f})$. In a similar way, one can prove that the term function of

$\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^{\#}(\hat{\alpha}, \hat{f})$ is equal to the term function of $(\hat{\Lambda}, \hat{E})^{\#} \circ {}^{\mathrm{t}}\hat{\mathcal{N}}(\hat{\alpha}, \hat{f})$. Since $\hat{\alpha} \in \Omega^1(\hat{S})$ and $\hat{f} \in C^{\infty}(\hat{S})$ are arbitrary, we obtain $\hat{\mathcal{N}} \circ (\hat{\Lambda}, \hat{E})^{\#} = (\hat{\Lambda}, \hat{E})^{\#} \circ {}^{\mathrm{t}}\hat{\mathcal{N}}$. Applying the same kind of technical arguments as before, we can deduce, after a hard computation, that $(\hat{\Lambda}, \hat{E})^{\#} \circ \mathcal{C}((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}}) = 0$. □

Remark 5.2. Under the assumptions of Theorem 5.1, if $(M, (\Lambda, E), \mathcal{N})$ is a Jacobi–Nijenhuis manifold which is reducible via (S, F) to $(\hat{S}, (\hat{\Lambda}_0, \hat{E}_0), \hat{\mathcal{N}})$, then, each member $(\hat{\Lambda}_k, \hat{E}_k)$ of the hierarchy $((\hat{\Lambda}_k, \hat{E}_k), k \in \mathbb{N})$ of Jacobi structures on $\hat{S}$, given by Theorem 4.3, is obtained by reduction via (S, F), from the corresponding member (Λ_k, E_k) of the hierarchy $((\Lambda_k, E_k), k \in \mathbb{N})$ of Jacobi structures on M.

Next proposition establishes a relation between reduction and conformal equivalence of Jacobi–Nijenhuis structures.

Proposition 5.3. *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis manifold, S a submanifold of M and F a vector sub-bundle of $T_S M$ which satisfy the conditions of Theorem* 2.1, *and $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$, $\hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, the Jacobi–Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (S, F). Let $a \in C^{\infty}(M)$ be a function which vanishes nowhere and such that* da *is a section of F^0, and $((\Lambda^a, E^a), \mathcal{N}^a)$ the Jacobi–Nijenhuis structure on M, a-conformal to $((\Lambda, E), \mathcal{N})$. Then $(M, (\Lambda^a, E^a), \mathcal{N}^a)$ is reducible via (S, F) and the reduced structure on $\hat{S}$ is conformally equivalent to $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$.*

Proof. Since da is a section of F^0, it is easy to check that if the Jacobi structure (Λ, E) on M is reducible via (S, F), then the a-conformal Jacobi structure (Λ^a, E^a) on M is also reducible via (S, F). Furthermore, condition 1 of Theorem 5.1 holds. So, $\hat{S}$ is equipped with two (reduced) Jacobi structures $(\hat{\Lambda}, \hat{E})$ and $(\widehat{\Lambda^a}, \widehat{E^a})$ that are compatible (cf. [12]). But $\widehat{\Lambda^a} = \hat{\Lambda}^{\hat{a}}$ and $\widehat{E^a} = \hat{E}^{\hat{a}}$, where $\hat{a} \in C^{\infty}(\hat{S})$ is given by $\hat{a} \circ \pi = a|_S$; i.e., the Jacobi structures $(\hat{\Lambda}, \hat{E})$ and $(\widehat{\Lambda^a}, \widehat{E^a})$ on $\hat{S}$ are conformally equivalent.

It remains to check that $\mathcal{N}^a := (N^a, Y^a, \gamma^a, g^a)$ verifies the conditions 2–5 of Theorem 5.1. Because Y is tangent to S and da vanishes on F, $N^a|_S(TS) \subset TS$ and $N^a|_S(F) \subset F$. Let $X \in \mathcal{V}^1(S)$ be a projectable vector field. Then, we have that $N_S^a(X) = N_S(X) - \langle \mathrm{d}a/a, X\rangle Y_S$ and, for any section Z of $TS \cap F$,

$$L_Z(N_S^a(X)) = L_Z(N_S(X)) - \left\langle \frac{\mathrm{d}a}{a}, X \right\rangle L_Z Y_S$$

is also a section of $TS \cap F$. So, $N_S^a(X) \in \mathcal{V}^1(S)$ and it is a projectable vector field. Also,

$$L_Z g^a = L_Z g + \left(L_Z \frac{1}{a}\right) L_Y a + \frac{1}{a} L_Z(L_Y a) = 0,$$

for all sections Z of $TS \cap F$, which implies that g^a is constant on the leaves of S. Finally, for the restriction $\gamma^a|_S$ of $\gamma^a \in \Omega^1(M)$ to the submanifold S, since ${}^{\mathrm{t}}N|_S(F^0) \subset F^0$, we obtain that $\gamma^a|_S$ is a section of $(TS \cap F)^0$ and that $i_Z\, d({}^{\mathrm{t}}Ti(\gamma^a|_S)) = 0$, for all sections Z of $TS \cap F$. From the definitions of $\mathcal{N}^a$ and $\hat{\mathcal{N}}$, it follows that $\widehat{\mathcal{N}^a} = \hat{\mathcal{N}}^{\hat{a}}$. □

Examples 5.4.

1. Let M be a five-dimensional C^∞-differentiable manifold equipped with a Jacobi–Nijenhuis structure $((\Lambda, E), \mathcal{N}), \mathcal{N} := (N, Y, \gamma, g)$, which is given, in local coordinates $(x_0, x_1, x_2, x_3, x_4)$, by

$$\Lambda = \frac{3}{2}\frac{\partial}{\partial x_0} \wedge \left(x_1\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_3}\right) + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}, \qquad E = \frac{3}{2}\frac{\partial}{\partial x_0},$$

$$\begin{aligned} N = {}& \left(-x_4\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) \otimes \mathrm{d}x_0 \\ & - \left(\left(x_4 + \frac{3}{2}x_1\right)\frac{\partial}{\partial x_1} + \frac{3}{2}(x_3 - x_2)\frac{\partial}{\partial x_3}\right) \otimes \mathrm{d}x_1 \\ & + \left(-x_4\frac{\partial}{\partial x_2} + \frac{5}{2}x_1\frac{\partial}{\partial x_3}\right) \otimes \mathrm{d}x_2 - x_4\frac{\partial}{\partial x_3} \otimes \mathrm{d}x_3 \\ & + \left(\frac{1}{2}x_0\frac{\partial}{\partial x_0} - x_1\frac{\partial}{\partial x_1} + \frac{3}{2}x_2\frac{\partial}{\partial x_2} + \frac{3}{2}x_3\frac{\partial}{\partial x_3} - x_4\frac{\partial}{\partial x_4}\right) \otimes \mathrm{d}x_4, \end{aligned}$$

$$Y = -\frac{3}{2}x_1^2\frac{\partial}{\partial x_1} + \left(\frac{1}{3}x_0 + \frac{3}{2}x_1(x_2 - x_3)\right)\frac{\partial}{\partial x_3},$$

$$\gamma = \frac{3}{2}(\mathrm{d}x_1 - \mathrm{d}x_4), \qquad g = \frac{3}{2}x_1 - x_4.$$

   If F denotes the vector sub-bundle of TM generated by the vector field $(\partial/\partial x_3)$, it is easy to check that all the conditions of Theorem 5.1 hold. So, $(M, (\Lambda, E), \mathcal{N})$ is reducible via (M, F) to a Jacobi–Nijenhuis manifold $(\hat{M}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}}), \hat{\mathcal{N}} := (\hat{N}, \hat{Y}, \hat{\gamma}, \hat{g})$, where

$$\hat{\Lambda} = \frac{3}{2}x_1\frac{\partial}{\partial x_0} \wedge \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, \qquad \hat{E} = \frac{3}{2}\frac{\partial}{\partial x_0},$$

$$\begin{aligned} \hat{N} = {}& \left(-x_4\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_2}\right) \otimes \mathrm{d}x_0 - \left(x_4 + \frac{3}{2}x_1\right)\frac{\partial}{\partial x_1} \otimes \mathrm{d}x_1 - x_4\frac{\partial}{\partial x_2} \otimes \mathrm{d}x_2 \\ & + \left(\frac{1}{2}x_0\frac{\partial}{\partial x_0} - x_1\frac{\partial}{\partial x_1} + \frac{3}{2}x_2\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_4}\right) \otimes \mathrm{d}x_4, \end{aligned}$$

$$\hat{Y} = -\frac{3}{2}x_1^2\frac{\partial}{\partial x_1}, \qquad \hat{\gamma} = \frac{3}{2}(\mathrm{d}x_1 - \mathrm{d}x_4), \qquad \hat{g} = \frac{3}{2}x_1 - x_4.$$

2. Let (M, Λ, N) be a Poisson–Nijenhuis manifold which is reducible via (S, F) to a Poisson–Nijenhuis manifold $(\hat{S}, \hat{\Lambda}, \hat{N})$ in the sense of Theorem 3.1, and let $a \in C^\infty(M)$ be a function that never vanishes. Then, $(M, (a\Lambda, \Lambda^\#(\mathrm{d}a)), \mathcal{N}), \mathcal{N} := (N, 0, {}^{\mathrm{t}}N(\mathrm{d}a/a), 0)$, is a Jacobi–Nijenhuis manifold. Moreover, if $a \in C^\infty(M)$ is in the conditions of Proposition 5.3, from the Poisson–Nijenhuis reduction assumptions on Λ and N, one can deduce that $(M, (a\Lambda, \Lambda^\#(\mathrm{d}a)), \mathcal{N})$ is reducible via (S, F) to the Jacobi–Nijenhuis manifold $(\hat{S}, (\hat{a}\hat{\Lambda}, \hat{\Lambda}^\#(\mathrm{d}\hat{a})), \hat{\mathcal{N}}), \hat{\mathcal{N}} := (\hat{N}, 0, {}^{\mathrm{t}}\hat{N}(\mathrm{d}\hat{a}/\hat{a}), 0)$, where $\hat{a} \in C^\infty(\hat{S})$ is given by $\hat{a} \circ \pi = a|_S$.

Now we are going to present the relationship between the reduction of a Jacobi–Nijenhuis manifold and the reduction of the corresponding homogeneous Poisson–Nijenhuis manifold, in the sense of Proposition 4.4.

Let $(M, (\Lambda, E), \mathcal{N})$ be a Jacobi–Nijenhuis manifold, S a submanifold of M, F a vector sub-bundle of $T_S M$, and $(\tilde{M}, \tilde{\Lambda}, \tilde{N}, \tilde{T})$ the corresponding homogeneous Poisson–Nijenhuis manifold, in the sense of Proposition 4.4. Consider the submanifold $\tilde{S} = S \times \mathbb{R}$ of $\tilde{M} = M \times \mathbb{R}$ and the vector sub-bundle $\tilde{F}$ of $T_{\tilde{S}}\tilde{M}$ given by $\tilde{F} = F \times \{0\}$. Then, $T\tilde{S} \cap \tilde{F} = (TS \cap F) \times \{0\}$. We denote by $\tilde{i} : \tilde{S} \hookrightarrow \tilde{M}$ the canonical injection and by $\tilde{\lambda} : T_{\tilde{S}}\tilde{M} \to T\tilde{S}$ a (projection) vector bundle map such that its restriction to $T\tilde{S}$ is the identity map and $\tilde{F} \subset Ker\,\tilde{\lambda}$. We should point out that the vector field $\tilde{T} = \partial/\partial t$ is tangent to $\tilde{S}$, $\tilde{T}|_{\tilde{S}} \notin Ker\,\tilde{\lambda}$ and $\tilde{\lambda}(\tilde{T}|_{\tilde{S}}) \in \mathcal{V}^1(\tilde{S})$ is a projectable vector field. Under these assumptions, we can state the following result.

Proposition 5.5. *If the homogeneous Poisson–Nijenhuis manifold* $(\tilde{M}, \tilde{\Lambda}, \tilde{N}, \tilde{T})$ *is reduced via* $(\tilde{S}, \tilde{F})$ *to a homogeneous Poisson–Nijenhuis manifold* $(\hat{\tilde{S}}, \hat{\tilde{\Lambda}}, \hat{\tilde{N}}, \hat{\tilde{T}})$, *then the Jacobi–Nijenhuis manifold* $(M, (\Lambda, E), \mathcal{N})$ *is reducible via* (S, F) *to a Jacobi–Nijenhuis manifold* $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$.

Moreover, $(\hat{\tilde{S}}, \hat{\tilde{\Lambda}}, \hat{\tilde{N}}, \hat{\tilde{T}})$ *is the homogeneous Poisson–Nijenhuis manifold that corresponds to* $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ *in the sense of Proposition* 4.4.

The following lemma is useful in the proof of Proposition 5.5.

Lemma 5.6. *A vector field* $\tilde{X} \in \mathcal{V}^1(\tilde{S})$ *is projectable by* $\tilde{\pi} : \tilde{S} \to \hat{\tilde{S}}$ *if and only if* $\tilde{X} = X + \tilde{f}(\partial/\partial t)$, *where* $X \in \mathcal{V}^1(S)$ *is projectable by* $\pi : S \to \hat{S}$ *and* $\tilde{f} \in C^\infty(\tilde{S})$ *is such that* $L_Z \tilde{f} = 0$, *for all sections* Z *of* $TS \cap F$.

Proof. Taking into account that a vector field $\tilde{X} \in \mathcal{V}^1(\tilde{S})$ can be written as $\tilde{X} = X + \tilde{f}(\partial/\partial t)$, with $X \in \mathcal{V}^1(S)$ and $\tilde{f} \in C^\infty(\tilde{S})$, and that a section of $T\tilde{S} \cap \tilde{F}$ can be identified with a section of $TS \cap F$, the conclusion follows readily. □

Proof (of Proposition 5.5). It is known (cf. [10]) that if the Poisson manifold $(\hat{\tilde{S}}, \hat{\tilde{\Lambda}})$ is obtained from $(\tilde{M}, \tilde{\Lambda})$ by reduction via $(\tilde{S}, \tilde{F})$, then the Jacobi manifold $(\hat{S}, \hat{\Lambda}, \hat{E})$ is obtained from (M, Λ, E) by reduction via (S, F) and, as a consequence of $T\tilde{S} \cap \tilde{F} = (TS \cap F) \times \{0\}$, $\hat{\tilde{S}} = \hat{S} \times \mathbb{R}$. Moreover, since $\tilde{F}^0 = F^0 \times T^*\mathbb{R}$, $(\tilde{\Lambda}|_{\tilde{S}})^\#(\tilde{F}^0) \subset T\tilde{S}$ implies $(\Lambda|_S)^\#(F^0) \subset TS$ and that $E|_S$ is a section of TS. From $\tilde{N}|_{\tilde{S}}(\tilde{F}) \subset \tilde{F}$, we obtain $N|_S(F) \subset F$ and also that $\gamma|_S$ is a section of $(TS \cap F)^0$, and from $\tilde{N}|_{\tilde{S}}(T\tilde{S}) \subset T\tilde{S}$, we get $N|_S(TS) \subset TS$ and we may conclude that Y is tangent to S. Let $X \in \mathcal{V}^1(S)$ be a projectable vector field. Using the fact that $\tilde{X} = X + \partial/\partial t \in \mathcal{V}^1(\tilde{S})$ is a projectable vector field and hence $\tilde{N}_{\tilde{S}}(\tilde{X}) = N_S(X) + Y_S + (\langle \gamma_S, X\rangle + g_S)\partial/\partial t$ is also a projectable vector field, from Lemma 5.6 we conclude that $N_S(X)$ and Y_S are projectable vector fields on S. In addition, $\tilde{N}_{\tilde{S}}(X) = N_S(X) + \langle {}^{\mathrm{t}}(Ti)(\gamma|_S), X\rangle(\partial/\partial t) \in \mathcal{V}^1(\tilde{S})$ is also a projectable vector field and from Lemma 5.6, for all sections Z of $TS \cap F$,

$$L_Z \langle {}^{\mathrm{t}}(Ti)(\gamma|_S), X\rangle = 0. \tag{34}$$

Since (34) holds for all projectable vector fields X on S, and taking into account that, for any $x \in S$, the projectable vector fields form a basis of $T_x S$, we deduce that $i_Z\, d({}^{\mathrm{t}}Ti(\gamma|_S)) = 0$, for all sections Z of $TS \cap F$. Finally, because $\tilde{N}_{\tilde{S}}(\partial/\partial t) = Y_S + g|_S(\partial/\partial t)$ is a projectable vector field on $\tilde{S}$, from Lemma 5.6 we have that $L_Z g|_S = 0$ for all sections Z of $TS \cap F$. Thus, we conclude that the Jacobi–Nijenhuis manifold $(\hat{S}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ is obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (S, F).

The last part of the proposition is a consequence of the fact that $T\tilde{\pi} = (T\pi, id_{T\mathbb{R}})$ and $\tilde{\lambda} = (\lambda, id_{T\mathbb{R}})$. □

6. Reduction under Lie group actions

Let ϕ be a left action of a Lie group G on a Jacobi manifold (M, Λ, E). ϕ is said to be a *Jacobi action* if, for all $h \in G$, the map $\phi_h : M \to M$, $\phi_h(x) = \phi(h, x)$, is a Jacobi diffeomorphism. The action ϕ is *proper* if the space $\hat{M}$ of the orbits has the structure of a differentiable manifold for which the canonical projection $\pi : M \to \hat{M}$ is a submersion.

Let $\mathcal{G}$ denote the Lie algebra of G. For any $X \in \mathcal{G}$, let $X_M \in \mathcal{V}^1(M)$ be the fundamental vector field corresponding to X,

$$X_M(x) = \frac{\mathrm{d}}{\mathrm{d}t}(\phi(\exp(-tX), x))|_{t=0}, \quad x \in M.$$

If the Lie group G is connected, then ϕ is a Jacobi action if and only if $[X_M, \Lambda] = 0$ and $[X_M, E] = 0$, for all $X \in \mathcal{G}$.

Proposition 6.1. *Let $(M, (\Lambda, E), \mathcal{N}), \mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis manifold, G a connected Lie group that acts on M with a proper Jacobi action ϕ and F the vector sub-bundle of TM tangent to the orbits of ϕ. If for all $X \in \mathcal{G}$, $L_{X_M} N = 0$, $L_{X_M} Y = 0$, $L_{X_M}\gamma = 0$, $i_{X_M}\gamma = 0$, $L_{X_M} g = 0$, and $N(X_M) = (\xi(X))_M$, where $\xi : \mathcal{G} \to \mathcal{G}$ is an endomorphism, then, the space $\hat{M}$ of the orbits of ϕ is endowed with a structure of a Jacobi–Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via (M, F).*

Proof. A straightforward calculation leads to the conclusion that all the conditions of Theorem 5.1 hold. □

Let us now suppose that the Jacobi action ϕ of the connected Lie group G on the Jacobi–Nijenhuis manifold $(M, (\Lambda, E), \mathcal{N})$ admits a momentum map J; i.e., a map $J : M \to \mathcal{G}^*$, where $\mathcal{G}^*$ is the dual space of the Lie algebra $\mathcal{G}$ of G, such that for all $X \in \mathcal{G}$, $X_M = \Lambda^{\#}(\mathrm{d}\langle J, X\rangle) + \langle J, X\rangle E$, where $\langle J, X\rangle \in C^\infty(M)$ is given by $\langle J, X\rangle(x) = \langle J(x), X\rangle$, for any $x \in M$. In addition, we suppose that J is Ad^*-equivariant, i.e., $J \circ \phi_h = Ad_h^* \circ J$, for all $h \in G$, where Ad^* is the coadjoint action of G on $\mathcal{G}^*$.

Let $\mu \in \mathcal{G}^*$ be a weakly regular value of J. Then, $S = J^{-1}(\mu)$ is a submanifold of M and $T_x J^{-1}(\mu) = Ker(T_x J)$, for all $x \in J^{-1}(\mu)$. Denote by F the vector sub-bundle of $T_S M$ given by

$$F = \{X_M - \langle \mu, X\rangle E, \quad X \in \mathcal{G}\}. \tag{35}$$

Then $F \cap T(J^{-1}(\mu)) = \{X_M - \langle \mu, X\rangle E,\ X \in \mathcal{G}_\mu\}$, where $\mathcal{G}_\mu$ is the Lie algebra of the isotropy group G_μ. In [11], we proved that $F \cap T(J^{-1}(\mu))$ is a completely integrable vector sub-bundle of $T(J^{-1}(\mu))$ and, if it has constant rank and defines a simple foliation of $J^{-1}(\mu)$, then $(\widehat{J^{-1}(\mu)}, \hat{\Lambda}, \hat{E})$ is a Jacobi manifold obtained from (M, Λ, E) by reduction via $(J^{-1}(\mu), F)$. In this reduction procedure, one verifies that $(\Lambda|_S)^{\#}(F^0) \subset TS$ and $E|_S$ is a section of TS.

Keeping the notations of the previous sections, we may establish the following result for Jacobi–Nijenhuis structures.

Proposition 6.2. *Let $(M, (\Lambda, E), \mathcal{N})$, $\mathcal{N} := (N, Y, \gamma, g)$, be a Jacobi–Nijenhuis manifold such that the vector field E is complete. Let G be a connected Lie group which acts on M with a left Jacobi action that admits an Ad*-equivariant momentum map J. Let $\mu \in \mathcal{G}^*$ be a weakly regular value of J, $S = J^{-1}(\mu)$, and F the vector sub-bundle of T_SM given by* (35). *Suppose that $TS \cap F$ has constant rank and defines a simple foliation of S and that the following conditions hold:*

1. $T_SJ \circ N|_S = T_SJ$;
2. $\forall X \in \mathcal{G}$, $N|_S(X_M - \langle \mu, X\rangle E) = (\xi(X))_M - \langle \mu, \xi(X)\rangle E$, *where* $\xi : \mathcal{G} \to \mathcal{G}$ *is an endomorphism*;
3. $\forall X \in \mathcal{G}_\mu$, $L_{X_M} N_S = 0$ *and* $L_E N_S = 0$;
4. Y *is tangent to* $S = J^{-1}(\mu)$, $L_E Y = 0$, *and* $L_{X_M} Y = 0$, *for all* $X \in \mathcal{G}_\mu$;
5. $i_E(\mathrm{d}\gamma_S) = 0$ *and, for all* $X \in \mathcal{G}_\mu$, $L_{X_M}\gamma_S = 0$ *and* $i_{X_M}(\mathrm{d}\gamma_S) = 0$;
6. $g|_S$ *is a first integral of* E *and of* X_M, *for all* $X \in \mathcal{G}_\mu$.

Under these conditions, $(\widehat{J^{-1}(\mu)}, (\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ is a Jacobi–Nijenhuis manifold obtained from $(M, (\Lambda, E), \mathcal{N})$ by reduction via $(J^{-1}(\mu), F)$.

Proof. An easy computation shows that the condition 2 of Theorem 5.1 follows from hypotheses 1–3. On the other hand, from 4–6 of Proposition 6.2, conditions 3–5 of Theorem 5.1 also hold. Taking into account the previous comments, the proof is concluded. □

As observed in [11], the vector sub-bundle $T(J^{-1}(\mu)) \cap F$ of $T(J^{-1}(\mu))$ is the tangent bundle to the orbits of the restriction to $G_\mu \times J^{-1}(\mu)$ of the action ϕ' of G_μ on M defined, for all $x \in M$ and $X \in \mathcal{G}_\mu$, by $\phi'(\exp(tX), x) = \phi(\exp(tX), \rho_{t\langle\mu,X\rangle}(x))$, where $(\rho_t)_{t\in\mathbb{R}}$ is the flow of the vector field E. Thus, the Jacobi–Nijenhuis structure $((\hat{\Lambda}, \hat{E}), \hat{\mathcal{N}})$ obtained in Proposition 6.2 is in fact defined on the space $J^{-1}(\mu)/G_\mu$ of the orbits of the action ϕ' of G_μ on $J^{-1}(\mu)$.

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